VI. Communications Elements Research

TELECOMMUNICATIONS DIVISION

A. Spacecraft Antenna Research: Antenna Tolerances, *D. Damlamayan*

1. Introduction

The actual performance of an antenna always falls short of ideal performance. The reason for this is that the actual aperture distribution deviates from the ideal one due to various errors which may be either systematic or random.

Systematic errors are those which, once the cause is known, may be calculated accurately. Errors due to the displacement of the feed from the focus of a parabolic reflector or a lens are of this type.

Random errors are due to unpredictable deviations of the antenna parameters from their design value. Departure of the surface of a parabolic reflector or of a lens from its true shape gives rise to random errors.

In this article, the effect of various errors on the gain of antennas representable by a field distribution over an aperture is discussed, with particular reference to parabolic reflectors and lenses. The related subject of pattern degradation has not been considered. The subject of antenna tolerances has been studied quite extensively in recent years. Some of the more interesting works are cited in Refs. 1–6.

The axial gain of a highly directional aperture antenna, radiating close to the $\theta = 0$ direction, whose aperture distribution is $F(x, y)$, can be written as

$$G_0 = \frac{4\pi}{\lambda^2} \frac{\left| \int F(\underline{x})\, dS \right|^2}{\int \left| F(\underline{x}) \right|^2 dS} \tag{1}$$

where the integration is carried over the aperture (xy) plane. This result is most simply and directly obtained by considering the plane wave spectrum representation of the aperture field.

As our interest lies mainly in reflector and lens antennas, it will be sufficient to assume only a pure phase error. If such a phase error $\delta(\underline{x})$ is introduced, the gain becomes

$$G = \frac{4\pi}{\lambda^2} \frac{\left| \int F(\underline{x})\, e^{i\delta(\underline{x})}\, dS \right|^2}{\int \left| F(\underline{x}) \right|^2 dS} \tag{2}$$

$\delta(\underline{x})$ will be a deterministic function when systematic errors are considered and a random function when we treat random errors.

2. Random Errors

Let us first consider the gain loss resulting from random phase errors on the aperture plane caused by the deviations of the antenna surface from the ideal. Following Ruze (Refs. 1 and 5), we assume the phase error $\delta(\underline{x})$ at any point on the aperture to be a random sample from a single gaussian distribution with zero mean and variance $\overline{\delta^2}$. It follows that the surface deviations are assumed to be distributed uniformly over the aperture.

The numerator of Eq. (2), which represents the power radiated axially, can be written as a double surface integral:

$$P = \iint F(\underline{x}_1) F^*(\underline{x}_2) \exp\{i[\delta(\underline{x}_1) - \delta(\underline{x}_2)]\} dS_1 dS_2 \tag{3}$$

Letting $\underline{x}_1 - \underline{x}_2 = \underline{\tau}$ as the vector difference between two integration points on the aperture, and $y(\underline{\tau}) = \delta(\underline{x}_1) - \delta(\underline{x}_2)$ as the phase difference of two points on the aperture spaced a distance $\underline{\tau}$ apart, we get

$$P = \iint F(\underline{x} + \underline{\tau}) F^*(\underline{x}) e^{iy(\underline{\tau})} dS_\tau dS_x \tag{4}$$

Defining now the aperture illumination auto-correlation function as

$$\phi(\underline{\tau}) = \frac{\int F(\underline{x} + \underline{\tau}) F^*(\underline{x}) dS_x}{\int \left| F(\underline{x}) \right|^2 dS_x} \tag{5}$$

we obtain for the gain the simple expression

$$G = \frac{4\pi}{\lambda^2} \int \phi(\underline{\tau}) e^{iy(\underline{\tau})} dS_\tau \tag{6}$$

Since $\delta(\underline{x})$ and hence $y(\underline{\tau})$ are random functions, the quantity of interest is the expected value of the gain $\overline{G}$. The correlation distance of the phase errors, c, is defined as that distance at which the phase errors become essentially independent. Then for $\tau >> c$, $y(\underline{\tau})$ becomes the difference of two independent normal random variables, hence it is also normally distributed with variance

$$\overline{y^2(\underline{\tau})} \xrightarrow{\tau >> c} 2\overline{\delta^2} \tag{7}$$

On the other hand, for $\tau = 0$ the mean square difference in phase error is obviously zero:

$$\overline{y^2(0)} = 0 \tag{8}$$

We now assume a convenient form for the variance function $\overline{y^2(\tau)}$, which fits these end conditions, such as

$$\overline{y^2(\tau)} = 2\overline{\delta^2}(1 - e^{-\tau^2/c^2}) \tag{9}$$

Another assumed form for the variance function with a similar smooth behavior does not significantly alter the result. Now it can be shown that

$$\overline{e^{iy(\tau)}} = \exp[-\overline{\delta^2}(1 - e^{-\tau^2/c^2})] \tag{10}$$

Hence

$$\overline{G} = \frac{4\pi}{\lambda^2} e^{-\overline{\delta^2}} \int \phi(\tau) \exp(\overline{\delta^2} e^{-\tau^2/c^2}) dS_\tau$$

$$= G_0 e^{-\overline{\delta^2}} + \frac{4\pi}{\lambda^2} e^{-\overline{\delta^2}} \sum_{n=1}^{\infty} \int \phi(\tau) \frac{(\overline{\delta^2})^n}{n!} e^{-n\tau^2/c^2} dS_\tau \tag{11}$$

where G_0 is the gain for no phase errors. Because of the exponential factor the integral essentially vanishes for $\tau > c$. Now $\phi(0) = 1$, and assuming that c is small compared to the aperture dimensions we may set $\phi(\tau) = 1$ in evaluating the integral. Extending the integration limit to infinity, we finally obtain

$$\overline{G} = G_0 e^{-\overline{\delta^2}} + \left(\frac{2\pi c}{\lambda}\right)^2 e^{-\overline{\delta^2}} \sum_{n=1}^{\infty} \frac{(\overline{\delta^2})^n}{n \cdot n!} \tag{12}$$

The reduction in axial gain is given by

$$\frac{G}{G_0} = e^{-\overline{\delta^2}} + \frac{1}{\eta}\left(\frac{2c}{D}\right)^2 e^{-\overline{\delta^2}} \sum_{n=1}^{\infty} \frac{(\overline{\delta^2})^n}{n \cdot n!} \tag{13}$$

where we have considered a circular aperture of diameter D and η is the aperture efficiency defined by

$$G_0 = \eta\left(\frac{\pi D}{\lambda}\right)^2$$

For small tolerance losses and for correlation regions c small compared to the aperture diameter D, we may write

$$G = G_0 e^{-\overline{\delta^2}} = \eta \left(\frac{\pi D}{\lambda}\right)^2 e^{-(2\pi/\lambda)^2 \overline{\xi^2}} \tag{14}$$

where $\overline{\xi^2}$ is the variance of optical path error. The maximum gain is achieved at the wavelength

$$\lambda_m = 2\pi\sqrt{\overline{\xi^2}} \tag{15}$$

Finally, the rms surface deviations ϵ must be related to the variance of phase or optical path errors for various antennas. For shallow reflectors (large f/D)

$$\sqrt{\overline{\xi^2}} = 2\epsilon \tag{16}$$

while for lenses of large f/D

$$\sqrt{\overline{\xi^2}} = \epsilon|n-1| \tag{17}$$

where n is the refractive index of the lens. Practical considerations such as low surface reflections require that $|n-1| \sim 0.5$. Then for the same small surface deviations, the fractional loss in gain in parabolic reflectors is about 4 times as great as in lenses. For the same surface deviation, the maximum gain for a lens will be achieved at a wavelength about 4 times smaller than for parabolic reflectors.

It is possible to take into account the curvature of the reflector or of the lens surface. Let the rms axial and normal surface deviations be denoted respectively as Δz and Δh for both reflectors and lenses. Then (Fig. 1)

$$PQ = \Delta z \qquad \text{and} \qquad PN = \Delta h \tag{18}$$

with

$$\frac{\Delta h}{\Delta z} = \cos\theta \tag{19}$$

In the case of a reflector, the rms optical path error is (Fig. 1a)

$$\begin{aligned}\sqrt{\overline{\xi^2}} = PP' + PR &= \Delta z + \Delta z \cos 2\theta \\ &= 2\Delta z \cos^2\theta\end{aligned} \tag{20}$$

Also, for a parabolic reflector we have ($\alpha = 2\theta$),

$$x = FP\sin 2\theta = \frac{2f}{1+\cos 2\theta}\sin 2\theta = 2f\tan\theta \tag{21}$$

Hence

$$\cos\theta = \frac{1}{\sqrt{1+\left(\frac{x}{2f}\right)^2}} \tag{22}$$

and

$$\begin{aligned}\sqrt{\overline{\xi^2}} &= \frac{2\Delta z}{1+\left(\frac{x}{2f}\right)^2} \\ &= \frac{2\Delta h}{\sqrt{1+\left(\frac{x}{2f}\right)^2}}\end{aligned} \tag{23}$$

The curvature must then have the effect of reducing the gain loss.

Considering the lens, the rms optical path error can be seen to be (Fig. 1b)

$$\begin{aligned}\sqrt{\overline{\xi^2}} &= |PP' - nP'R| \\ &= PP'|1 - n\cos\alpha| \\ &= \frac{\Delta h}{\cos(\alpha+\theta)}|1-n\cos\alpha|\end{aligned} \tag{24}$$

On the other hand, for a lens,

$$x = FP\sin\alpha = \frac{(n-1)f}{n\cos\alpha - 1}\sin\alpha \tag{25}$$

The last two relations together with Snell's law, $\sin(\alpha+\theta) = n\sin\theta$, enable one to obtain an expression for $\sqrt{\overline{\xi^2}}$ involving only x/f, n and either Δh or Δz.

It is easy to see that these expressions for the rms optical path errors in reflectors and lenses reduce to the ones given above for large f/D. In fact, for large f/D, $\alpha \approx 0$, $\theta \approx 0$ and we immediately obtain

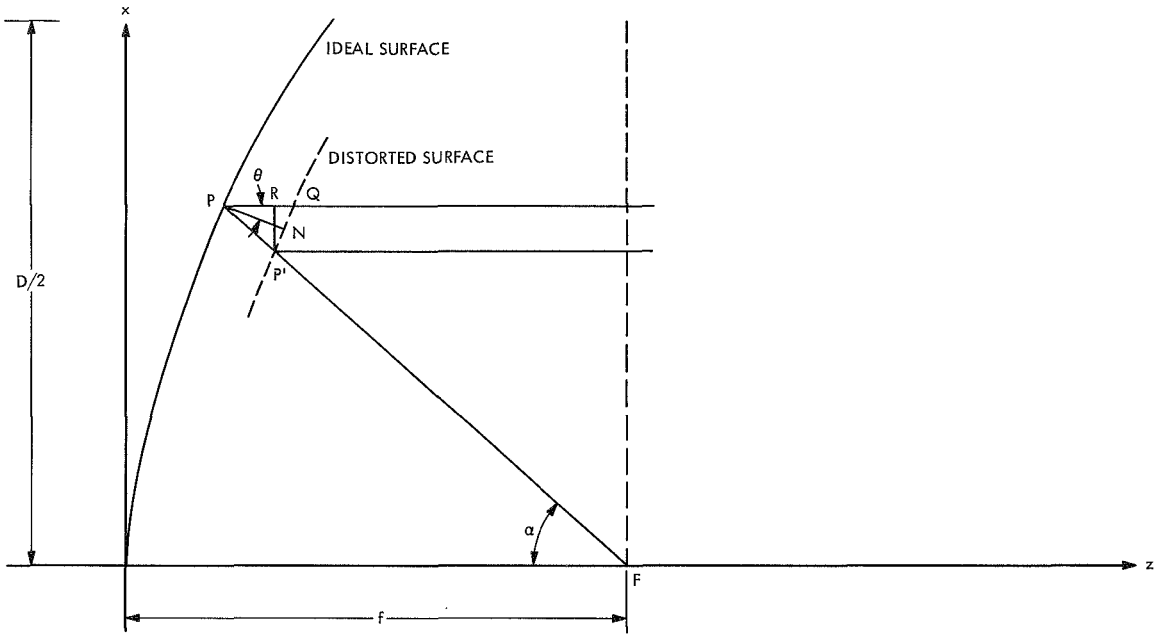
$$\sqrt{\overline{\xi^2}} = 2\Delta z = 2\Delta h \tag{26}$$

for reflectors, and

$$\sqrt{\overline{\xi^2}} = \Delta h|1-n| = \Delta z|1-n| \tag{27}$$

for lenses.

(a) PARABOLIC REFLECTOR



(b) LENS

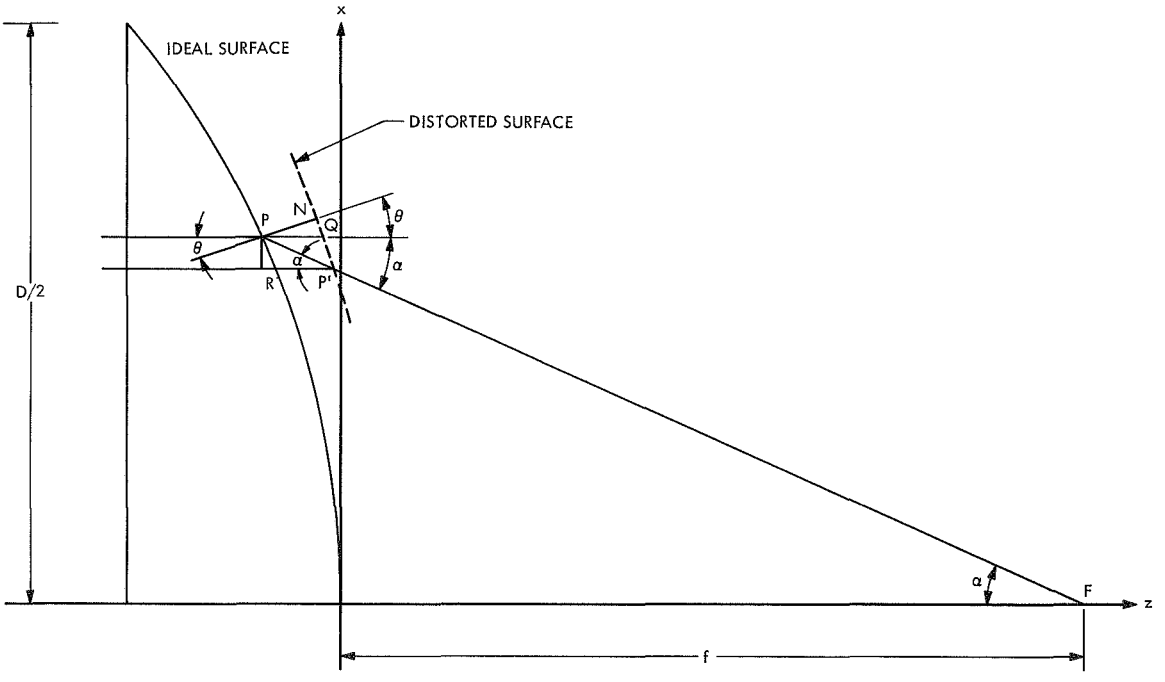


Fig. 1. Optical path length errors

In this connection, an extension of the above theory to surface deviations which are not distributed uniformly over the aperture must also be mentioned (Ref. 6). It is found that the assumption of uniform error distribution in general gives a lower value for the axial gain of an antenna whose surface deviations have regional variations over the aperture.

3. Systematic Errors

We next consider systematic errors due to defocussing of the source. The ratio of the gain to no error gain can be written as

$$\frac{G}{G_0} = \frac{\left| \int F(\underline{x})\, e^{i\delta(\underline{x})}\, dS \right|^2}{\left| \int F(\underline{x})\, dS \right|^2} \tag{28}$$

For convenience the phase error will be measured with respect to the illumination weighted mean phase error, which we call $\delta_0(\underline{x})$. Then

$$\delta_0(\underline{x}) = \delta(\underline{x}) - \frac{\int F(\underline{x})\, \delta(\underline{x})\, dS}{\int F(\underline{x})\, dS} \tag{29}$$

Assuming in-phase illumination and small phase errors, expanding the exponential in a power series, we obtain

$$\frac{G}{G_0} = \frac{\left| \int F(\underline{x}) \exp\left[i\delta_0(\underline{x})\right] dS \right|^2}{\left| \int F(\underline{x})\, dS \right|^2} = 1 - \overline{\delta_0^2} \tag{30}$$

where

$$\overline{\delta_0^2} = \frac{\int F(\underline{x})\, \delta_0^2(\underline{x})\, dS}{\int F(\underline{x})\, dS} \tag{31}$$

This well-known simple result is very useful in calculating the gain loss in various circumstances. For small phase errors, Eq. (30) is similar to Eq. (13) or (14). There are a number of differences, however, which should not be ignored. First, Eq. (13) is not restricted to small phase errors, and an extension of the theory that led to Eq. (13) is able to give information on the radiation pattern as well. Then Eq. (30) depends on the illumination function $F(\underline{x})$ whereas Eq. (13) does not. This is only so because in the statistical analysis the phase errors are assumed to be uniformly distributed. In this case $\overline{\delta_0^2}$ as given by Eq. (31) would also be independent of the illumination function.

The loss in gain incurred as a result of the feed displacement axially by a distance a from the focus F to a point F' is considered next. The optical path difference between an axial ray and one reaching the aperture at the point x is the same for both lens and parabolic reflector antennas of focal length f. Referring to Fig. 2 and noting that n is the refracting index for the lens and equals unity for the parabolic reflector, we have for both cases

$$\begin{aligned} \xi &= F'N - (F'O + nOP) \\ &\approx (FN - FR) - (F'O - nOP) \\ &= (FO + nOP - FR) - (F'O + nOP) \\ &= a(1 - \cos\alpha) \end{aligned} \tag{32}$$

Now for shallow reflectors we write

$$\cos\alpha = \frac{1 - \left(\frac{x}{2f}\right)^2}{1 + \left(\frac{x}{2f}\right)^2} = 1 - \frac{1}{2}\left(\frac{x}{f}\right)^2 + \frac{1}{8}\left(\frac{x}{f}\right)^4 + \cdots \tag{33}$$

with a similar expression for lenses. Hence, in either case, lumping the constants together, we write

$$\delta(x) = \frac{2\pi}{\lambda}\xi(x) = c_2 x^2 + c_4 x^4 \tag{34}$$

Considering only the quadratic term and a uniform illumination, we obtain

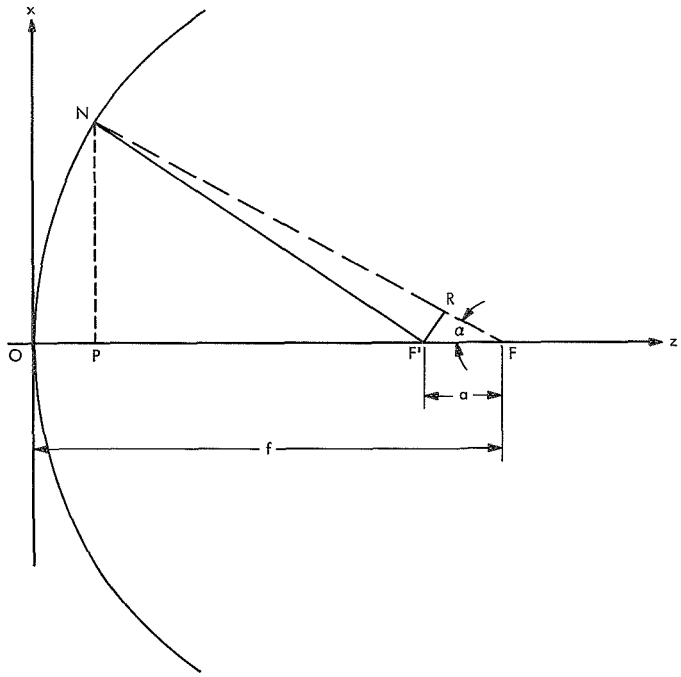
$$\delta_0(x) = c_2 x^2 - \frac{\Delta}{3} \tag{35}$$

where Δ is the phase error at the edge of the aperture. Then

$$\frac{G}{G_0} = 1 - \frac{1}{A}\int_A \left(c_2 x^2 - \frac{\Delta}{3}\right)^2 dA = 1 - \frac{4}{45}\Delta^2 \tag{36}$$

Thus in the case of uniform illumination an extreme phase error of about 1.1 radians can be tolerated if the loss in

(a) PARABOLIC REFLECTOR



(b) LENS

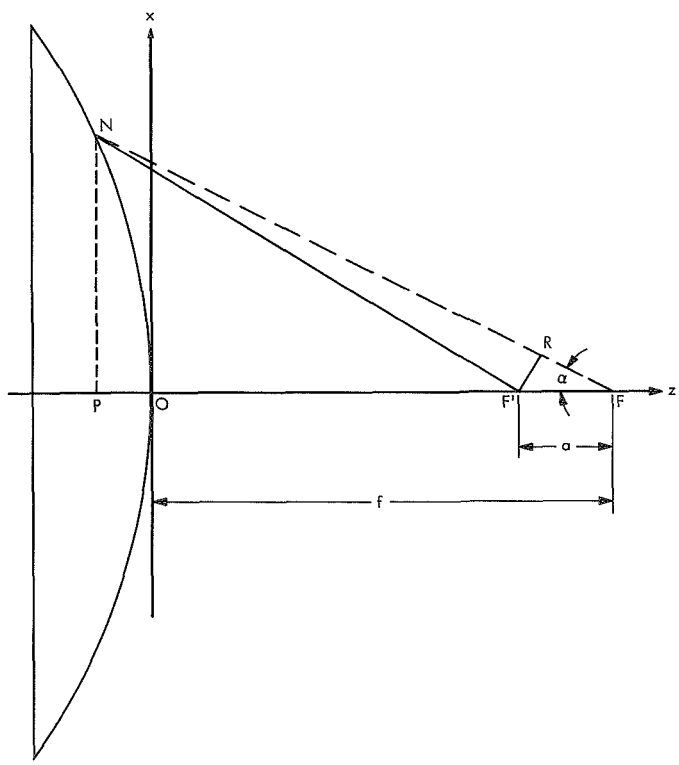


Fig. 2. Axial displacement of the feed from the focus

directivity is not to exceed 10%. This calculation for shallow reflectors is not a realistic one for communication antennas which may have small values of f/D ratios and is intended merely as an illustration.

The loss of gain resulting from the movement of the feed point away from the focus in a direction perpendicular to the axis can be calculated in a similar way. In this case, for small phase errors across the aperture, we can write

$$\delta(x) = c_1 x + c_3 x^3 \tag{37}$$

The constants may be calculated from the geometry for lenses and parabolic reflectors. The linear term causes the radiation pattern to tilt by an angle $c_1\lambda/2\pi$ without change of shape and hence has no bearing on the loss of gain. Only the cubic term contributes to the reduction in gain which, however, is small since the cubic errors are strong near the edge of the aperture where usually the illumination is reduced.

It must be mentioned in this connection that the two surfaces of a lens can be chosen so as to minimize the loss of gain resulting from the feed movement, an option which does not exist for parabolic reflectors, and which makes lenses particularly suitable for wide-angle scanning.

Finally, the effect of systematically misplaced plates in metal-plate lenses will be considered. If the theoretical distance between the plates is s and the actual distance is $s + \Delta s$, where $|\Delta s| << s$, the index of refraction for the theoretical situation and the change in the index are

$$n = \sqrt{1 - \left(\frac{\lambda}{2s}\right)^2} \tag{38}$$

and

$$\Delta n = \frac{\partial n}{\partial s}\Delta s \tag{39}$$

Then the phase error becomes

$$\begin{aligned}\delta(x) &= \frac{2\pi}{\lambda} z(x)\,\Delta n \\ &= \frac{\pi\lambda}{n} z(x)\frac{\Delta s}{2s^3} \\ &= \frac{4\pi}{\lambda}\frac{z(x)}{\lambda}\frac{(1-n^2)^{3/2}}{n}\Delta s\end{aligned} \tag{40}$$

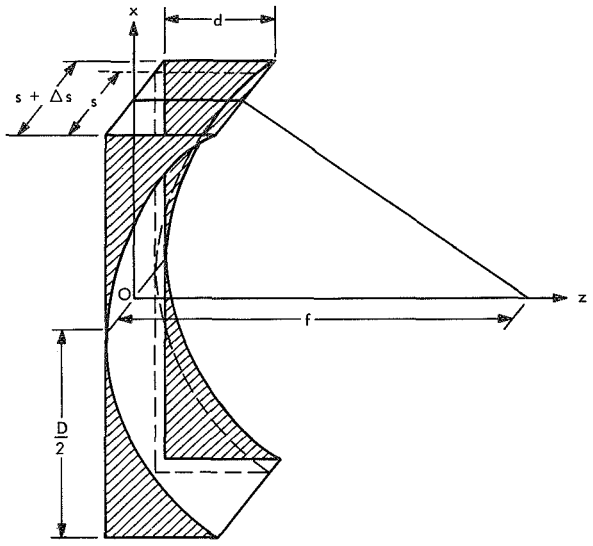


Fig. 3. Incorrect plate location in metal-plate lenses

The equation of the surface $z(x)$ can be found, referring to Fig. 3, from the relation

$$f = nz + \sqrt{x^2 + (f - z)^2}$$

which, upon squaring, becomes

$$(1 - n^2)z^2 - 2f(1 - n)z + x^2 = 0 \qquad (41)$$

These results also apply for random spacings, where the spacings s form a random sample from a single gaussian distribution with zero mean and rms deviation Δs. The gain loss may now be calculated from the expression for $\delta(x)$. We see that increasing the index of refraction or decreasing the thickness of the lens lowers the phase error. Thus reducing the thickness of a lens by stepping it helps to ease this tolerance, although, in general, stepping a lens causes a loss in gain because of radiation scattered in "shadow" areas.

As an example consider a metal-plate lens of large f/D ratio. In this case,

$$z(x) \approx \frac{f}{2(1 - n)}\left(\frac{x}{f}\right)^2 \qquad (42)$$

and a quadratic phase error is introduced. For a uniform aperture distribution and a quadratic phase error $\overline{\delta_0^2}$ has already been calculated. If the lens is stepped such that its thickness never exceeds $\lambda/(1 - n)$, we obtain

$$\overline{\delta_0^2} \leq \left(\frac{4\pi}{\lambda}\right)^2 \frac{1}{2}\Delta s^2 \qquad (43)$$

for $0.4 \leq n \leq 0.6$.

4. Conclusion

It has been shown quantitatively that the manufacturing tolerances for lenses can be less stringent than for reflectors. This is still so even if the possibility of errors arising from plate-spacing tolerances are considered. Easier tolerances and capability for better scanning are, in fact, the main advantages of lenses over reflectors.

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